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COMMENT

The Gelfand–Levitan–Marchenko equation and Bäcklund transformation

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Abstract. We have shown that with the help of the Gelfand-Levitan-Marchenko equation it is possible to deduce Bäcklund transformations in various situations. Firstly we discuss the case of explicitly 'x' and 't' dependent KdV equation as examples. It is also demonstrated that different forms of Bäcklund transformations arise for the different structure of the Gelfand-Levitan-Marchenko input kernel. The BT derived is also invertible. We have demonstrated that it is also possible actually to solve the BT starting with the special solution $q = 0$, though in algebraic structure the form of the BT is very complicated. The 't' dependent parts of the BTs are also deduced. Lastly, it is shown that our method also works with the matrix inverse scattering problem.

During the last few years various methods have been proposed for the derivation of the Bäcklund transformation (BT) associated with any nonlinear equation. We have shown here that it is possible to construct the BT in a straightforward way starting from the Gelfand-Levitan-Marchenko (GLM) equation, even in the non-autonomous cases, where the nonlinear equation contains explicit x and t dependence.

Let us consider the equation

$$q_t + 6qq_x + q_{xxx} + (r/2t)q = 0 \tag{1}$$

known as the cylindrical KdV equation. The space part of the Lax Equation is

$$-\Psi_{xx} + (q(x, t) + x)\Psi = -\lambda\Psi \tag{2}$$

which even for asymptotically falling solutions is not a free Schrödinger equation. But very recently a spectral analysis has been performed for equation (1) (previously some results were also obtained by Calogero *et al* 1980) from which we can conclude that the kernel in this case can also be written as

$$F(x, y) = -\Sigma c_i g_1^i(x) g_2^i(y) \tag{3}$$

where g_j^i stands for an eigenfunction and the subscript 1 or 2 indicates the particular 'potential' or nonlinear field; i is the superscript to the ith eigenvalue.

Then the solution of the GLM Equation

$$F(x, y) + K(x, y) + \int F(x, z)K(z, y)\,dz = 0 \tag{4}$$

can be put into the form

$$K(x, y) = -\sum d_j g_2^j(x) g_1^j(y) \tag{5}$$

then the difference $q_2 - q_1$ is given by

$$q_2 - q_1 = \Delta q = \mathrm{d}K(x, x)/\mathrm{d}x. \tag{6}$$

Now the Bäcklund transformation is a direct connection between the old and new fields q_1 and q_2 which are governed by the same nonlinear equation. In our formalism this is obtained by eliminating the eigenfunctions g_1^i and g_2^i between (3), (5) and (4). Here it is worthwhile mentioning that, before the elimination is done, one must specify the number of terms that must be retained in equations (3) and (5). Let us demonstrate with the help of only one term in (3) and (5). That is

$$K = -c_1 g_1(x) g_2(y), \qquad F = -d_1 g_1(x) g_1(y). \tag{7}$$

In our present discussions we restrict ourselves only to the bound state spectrum of the potential. Differentiating equation (6) we get

$$\Delta q = -c_1(g_1 g_2' + g_1' g_2)$$

$$(\Delta q)' = -c[(q_1 + q_2) g_1 g_2 - 2\lambda^2 g_1 g_2 + 2g_1' g_2' + 2x g_1 g_2]$$

$$(\Delta q)'' = (q_1 + q_2)K + (q_1 + q_2)\Delta q - 4\lambda^2 \Delta q + 2K + 4x\Delta q - 2cq_1 g_1 g_2' - 2cq_2 g_2 g_1'. \tag{8}$$

From equations (8), (7) and (5) we have

$$(g_1'/g_1) = (2\Delta q)^{-1}[(\Delta q)''/K + 4\lambda^2 \Delta q/K - (q_2 + 3q_1 + 4x)\Delta q/K - (q_2 + q_1)' - 2]. \tag{9}$$

However, on the other hand g_1'/g_1 satisfies the Riccati equation:

$$(\mathrm{d}/\mathrm{d}x)(g_1'/g_1) + (g_1'/g_1)^2 = (\lambda^2 - q_1 - x). \tag{10}$$

Therefore substituting from (10) for g_1'/g_1, we get a differential relation between q_2 and q_1 as follows

$$\begin{aligned} 2[(\Delta q)^2 + (\Delta q)'K]\{-[(\Delta q)'' + 4\lambda^2 \Delta q - (q_2 + 3q_1 + 4x)\Delta q - K(q_2 + q_1)' - 2K]\} \\ + 2K(\Delta q)[(\Delta q)''' + 4\lambda^2(\Delta q)' - (q_2 + 3q_1 + 4x)'\Delta q - (q_2 + 3q_1 + x)\Delta q'] \\ = 4(\lambda^2 - q' - x)(\Delta q)^2 K^2 + 2K(\Delta q)[(q_2 + q_1)''K + (q_2 + q_1)K' + 2K'] \\ - [(\Delta q)'' + 4\lambda^2(\Delta q) - (q_2 + 3q_1 + 4x)\Delta q - (q_2 + q_1 + 2)K]^2 \end{aligned} \tag{11}$$

which is nothing other than a relation between the nonlinear fields q_1 and q_2 and is the Bäcklund transformation. A similar calculation can also be performed for the equation

$$V_t = 6VV_x - 12V_x/x^2 + 24V/x^3 + V_{xxx} = 0 \tag{12}$$

which has been discussed by Ablowitz and Cornille (1979). In this case the eigenfunctions satisfy

$$-\Psi_{xx} + q_{(i)}\Psi + (\alpha/x^2)\Psi - K^2\Psi = 0, \qquad i = 1, 2, \ldots \tag{13}$$

and a similar calculation yields

$$\begin{aligned} 2K(\Delta q)\{(\Delta q)''' + 4\lambda^2(\Delta q)' - (\Delta q)'(q_2 + q_1 + 4\alpha/x^2) - (q_2 + q_1)'\Delta q + (8\alpha/x^3)\Delta q \\ + [4\alpha/x^3 - (q_2 + q_1)]K' - (12\alpha/x^4)K - K(q_2 + q_1)''\} \end{aligned}$$

$$
\begin{aligned}
&-2(\Delta q K'+K(\Delta q)') \\
&\times\{(\Delta q)''+4\lambda^2\Delta q-(q_2+3q_1+4\alpha/x^2)\Delta q+[4\alpha/x^3-(q_2+q_1)K]\} \\
&+\{(\Delta q)''+4\lambda^2\Delta q-(q_2+3q_1+4\alpha/x^2)\Delta q+[4\alpha/x^3-(q_2+q_1)']K\}^2 \\
&=4K^2(q_1+\alpha/x^2-\lambda^2)(\Delta q)^2.
\end{aligned}
\tag{14}
$$

Equations (11) and (14) yield the space part of the Bäcklund transformation. The time part can be deduced by utilising the temporal equation of the eigenfunction g_1^i and g_2^i which are of the form

$$
\begin{aligned}
&g_{1t}=-4g_{1xxxx}+6q_1g_{1x}+3q_{1x}g_1 \\
&g_{2t}=-4g_{2xxx}+6q_2g_{2x}+3q_{2x}g_2.
\end{aligned}
\tag{15}
$$

Now again from equation (8)

$$\Delta q=(g_2g_1'+g_1g_2').$$

Therefore

$$(\Delta q)_t=g_1'g_{2t}+g_2g_{1t}'+g_1g_{2t}'+g_{1t}g_2'. \tag{16}$$

The Riccati equation for the temporal evolution of g_1'/g_1 is

$$
\begin{aligned}
&(\mathrm{d}/\mathrm{d}t)(g_1'/g_1)+(2q_1+4x-4\lambda)(g_1'/g_1)^2+(8-2q_1'-4)(g_1'/g_1) \\
&\qquad=(2q_1-4\lambda-4x)(\lambda+q_1+x)-q_1''.
\end{aligned}
\tag{17}
$$

In which we can use the expression for g_1'/g_1 given by equation (9) to get the time part of the Bäcklund transformation (BT).

One of the most important properties of any BT is that if we can pass from q_1 to q_2 then it will also be possible to go back from q_2 to q_1, that is the BT is invertible, because the same calculation can be done with g_2'/g_2 instead of g_1'/g_1 and we can obtain the opposite BT obtaining q_1 from q_2. Two other important properties associated with a BT are (a) the generation of a non-trivial solution from a trivial one, (b) the BT itself can be thought of as a contact transformation. In the case of our BT given by equation (11) we can prove that

$$
\begin{aligned}
(g_1'/g_1)-(g_2'/g_2)&=(\Delta q)''/\Delta qK+4\lambda^2/K \\
&\quad-(2q_2+2q_1+2x)(1/K)-(q_2+q_1)'/\Delta q-2/\Delta q
\end{aligned}
\tag{18}
$$

and

$$(g_1'/g_1)+(g_2'/g_2)=K'/K$$

and also

$$(\mathrm{d}/\mathrm{d}x)(g_1'/g_1-g_2'/g_2)+[(g_2'/g_1)^2-(g_2'/g_2)^2]=\Delta q=K'. \tag{19}$$

Substituting from (9) and (18) in (19), we get after a rather long calculation (where we have set $q_1=0$)

$$K'''/K'-K''^2/K'^2+2K'/K+2xK''/K-2xK'^2/K^2=0$$

which can be integrated to

$$K''=2xK'^2/K$$

or

$$(\partial/\partial x)(K'/K)=(2x-1)K'^2/K^2$$

or

$$K'/K = 1/(\alpha - x^2 + x)$$

or

$$\log K = -\int \frac{dx}{x^2 - x - \alpha}$$

or

$$\therefore \quad K = [(x-\tfrac{1}{2}) - \beta]/[(x-\tfrac{1}{2}) + \beta] \qquad \text{with } \beta = (\alpha + \tfrac{1}{4})^{1/2}. \tag{20}$$

Since $q_1 = 0$ we get from equation (20) and (6)

$$q_2 = \frac{(x - \frac{1}{2} + \beta) - [(x - \frac{1}{2}) - \beta]}{[(x - \frac{1}{2}) + \beta]^2} = \frac{2\beta}{[x + (\beta - \frac{1}{2})]^2} \tag{21}$$

which is nothing other than a rational solution.

Let us now consider the case with more than one eigenfunction in the kernel function F and K. Let us define,

$$y = g_1' g_2, \qquad z = g_1 g_2', \qquad \omega = g_1 g_2, \qquad t = g_1' g_2', \tag{22}$$

from which we can deduce that,

$$\begin{pmatrix} y \\ z \\ \omega \\ t \end{pmatrix}_x = \begin{bmatrix} 0 & 0 & q_2 - \xi^2 & 1 \\ 0 & 0 & q_1 - \xi^2 & 1 \\ 1 & 1 & 0 & 0 \\ q_1 - \xi_2 & q_2 - \xi^2 & 0 & 0 \end{bmatrix} \begin{pmatrix} y \\ z \\ \omega \\ t \end{pmatrix}. \tag{23}$$

Now the kernel is written as:

$$K = g_1^2 g_1' + g_2^2 g_2'. \tag{24}$$

In the same way as before we go on computing (Δq), $(\Delta q)'$, $(\Delta q)''$ . . . etc., from these expressions. We can eliminate the $g_1^2 g_1'$ and all its derivatives to obtain the relationship between q_2 and q_1. The result is extremely complicated and too elaborate to be reproduced here.

Let us now demonstrate the effectiveness of our method for the matrix inverse scattering transform (IST) and matrix GLM equation.

Let the matrix eigenvalue problem be

$$(\mathrm{i}\sigma_3\, d/dx + Q_i)\Psi_i = \lambda \Psi_i; \qquad Q_i = \begin{pmatrix} 0 & q_i \\ p_i & 0 \end{pmatrix}. \tag{25}$$

Then it is known that

$$\mathrm{i}\sigma_3 K_x(x, y) + \mathrm{i}K_y(x, y)\sigma_3 + Q(x)K(x, y) - K(x, y)Q_0(y) = 0 \tag{25}$$

along with

$$[\sigma_3, K(x, x)] = -\mathrm{i}[Q(x) - Q_0(x)]. \tag{27}$$

Let us denote

$$\int_x^{\alpha} dy [p(x)q(y) - p_0(x)q_0(y)] = I$$

then it is easy to prove

$$K=-\tfrac{1}{2}\mathrm{i}(Q-Q_0)+\tfrac{1}{2}I. \tag{28}$$

Let us assume that we have two sets of eigenvalues to start with i.e. corresponding to the situation of the two-soliton solution, we set

$$K=\mathrm{i}c_1\Psi_1\Psi_0^T\sigma_1+\mathrm{i}\sigma_2\phi_1\phi_0^T\sigma_1 \tag{29}$$

where ϕ_1 and ϕ_0 are solutions of equation (25) with different eigenvalues. Then we have, differentiating (27),

$$\begin{aligned}\tfrac{1}{2}(\mathrm{d}/\mathrm{d}x)(Q-Q_0)&=\tfrac{1}{2}\mathrm{i}(\mathrm{d}/\mathrm{d}x)I\\&=\mathrm{i}\sigma_3(\lambda-Q)K+K\sigma_1(\lambda-Q_0^T)\sigma_3\sigma_1+\mathrm{i}\sigma_2(\lambda_1-\lambda)[\sigma_3\phi_1\phi_0^T\sigma_1+\phi_1\phi_0^T\sigma_3\sigma_1]\end{aligned}$$

now we can solve for $\phi_1\phi_0^T$ as

$$\phi_1\phi_0^T=\frac{-\mathrm{i}\sigma_3}{c_2(\lambda_1-\lambda)}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(Q-Q_0)-\mathrm{i}\tfrac{1}{2}I_x-\mathrm{i}\sigma_3(\lambda-Q)K-K\sigma_1(\lambda-Q_0^T)\sigma_3\sigma_1\right)\sigma_1 \tag{30}$$

also we have

$$\begin{aligned}\mathrm{i}c_1\Psi_1\Psi_0^T\sigma_1=K-(\lambda_1-\lambda)^{-1}\{&\sigma_3[(\mathrm{d}/\mathrm{d}x)(Q-Q_0)-\tfrac{1}{2}(\mathrm{d}/\mathrm{d}x)I\\&-\mathrm{i}\sigma_3(\lambda-Q)K-K\sigma_1(\lambda-Q_0^T)]\sigma_3\sigma_1\}.\end{aligned} \tag{31}$$

Now differentiating equation (29) with respect to x again we have, using equation (30) and (28)

$$\begin{aligned}\tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2)(Q-Q_0)&-\tfrac{1}{2}\mathrm{i}(\mathrm{d}^2/\mathrm{d}x^2)I\\&\equiv(\lambda^2-Q_2)(K-\mathrm{i}\sigma_3M\sigma_1)+2\sigma_3(\lambda-Q)(K\sigma_1-\mathrm{i}c_2M)\\&\quad\times(\lambda-Q_0^T)\sigma_3\sigma_1+(K-\mathrm{i}c_2M\sigma_1)(\lambda^2-Q_0^{T2})\sigma_3\sigma_1\\&\quad+\mathrm{i}c_2M\sigma_1(\lambda_1^2-Q^2)+2\mathrm{i}c_2\sigma_3(\lambda_1-Q)M(\lambda_1-Q_0^T)\sigma_3\sigma_1+\mathrm{i}c_2M(\lambda_1^2-Q_0^{T2})\sigma_1\end{aligned}$$

which is the direct relation between the two sets of nonlinear fields Q and Q_0 via the solution for K given by equation (28). So we call this the generalised BT.

In the above we have discussed a simple method of deducing the BT which utilises only the machinery of inverse scattering equations—and nothing else—like the discrete symmetry, of the nonlinear equation or of the IST equations. The derived BT has the property of invertibility and generates solutions from the trivial one (for which we have considered only the space part, as in the usual case). Although the form of BT obtained is more complicated than the usual one, the straightforward method of obtaining it makes it worth studying.

References

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